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Hybrid Feedback Stabilization of Nonlinear Systems with Quantization Noise and Large Delays

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Summary. Control systems over networks with a finite data rate can be conveniently modeled as hybrid (impulsive) systems. For the class of nonlinear systems in feedforward form, we design a hybrid controller which guarantees stability, in spite of the measurement noise due to the quantization, *and* of an arbitrarily large delay which affects the communication channel. The rate at which feedback packets are transmitted from the sensors to the actuators is shown to be arbitrarily close to the infimal one.

This paper is dedicated to Professor Alberto Isidori on the occasion of his 65th birthday, with admiration and gratitude.

1 Introduction

The problem of controlling systems under communication constraints has attracted much interest in recent years. In particular, many papers have investigated how to cope with the finite bandwidth of the communication channel in the feedback loop. For the case of linear systems (cf. e.g. [2, 7, 9, 8, 23, 27, 3]) the problem has been very well understood, and an elegant characterization of the minimal data rate – that is the minimal rate at which the measured information must be transmitted to the actuators – above which stabilization is always possible is available. Loosely speaking, the result shows that the minimal data rate is proportional to the inverse of the product of the unstable eigenvalues of the dynamic matrix of the system. Controlling using the minimal data rate is interesting not only from a theoretical point of view, but also from a practical one, even in the presence of communication channels with a large bandwidth. Indeed, having control techniques which employ a small number of bits to encode the feedback information implies for instance that the number of different tasks which can be simultaneously carried out is maximized, results in explicit procedures to convert the analog information provided by the sensors into the digital form which can be transmitted, and improves the performance of the system ([15]). We refer the reader to [25] for an excellent survey on the topic of control under data rate constraints.

The problem for nonlinear systems has been investigated as well (cf. [16, 18, 24, 6, 13, 4]). In [16], the author extends the results of [2] on quantized control to nonlinear systems which are *input-to-state* stabilizable. For the same class, the paper [18] shows that the approach in [27] can be employed also for continuous-time nonlinear systems, although in [18] no attention is paid on the minimal data rate needed to achieve the result. In fact, if the requirement on the data rate is not strict, as it is implicitly assumed in [18], it is shown in [6] that the results of [18] actually hold for the much broader class of *stabilizable* systems. The paper [24] shows, among the other results, that a minimal data rate theorem for *local* stabilizability of nonlinear systems can be proven by focusing on linearized system. To the best of our knowledge, *non* local results for the problem of minimal data rate stabilization of nonlinear systems are basically missing. Nevertheless, the paper [4] has pointed out that, if one restricts the attention to the class of nonlinear *feedforward* systems, then it is possible to find the infimal data rate above which stabilizability is possible. We recall that feedforward systems represent a very important class of nonlinear systems, which has received much attention in recent years (see e.g. [29, 22, 12, 10, 19], to cite a few), in which many physical systems fall ([11]), and for which it is possible to design stabilizing control laws in spite of saturation on the actuators. When *no* communication channel is present in the feedback loop, a recent paper ([20], see also [21]) has shown that any feedforward nonlinear system can be stabilized regardless of an arbitrarily large delay affecting the control action.

In this contribution, exploiting the results of [20], we show that the minimal data rate theorem of [4] holds when an arbitrarily large delay affects the channel (in [4], instantaneous delivery through the channel of the feedback packets was assumed). Note that the communication channel not only introduces a delay, but also a quantization error and an impulsive behavior [26], since the packets of bits containing the feedback information are sent only at discrete times. Hence, the methods of [20], which are studied for continuous-time delay systems, can not be directly used to deal with impulsive delay systems in the presence of measurement errors. In addition, our result requires an appropriate redesign, not only of the parameters in the feedback law of [20], but also of the encoder and the decoder of [4]. See [17] for another approach to control problems in the presence of delays and quantization.

In the next section, we present some preliminary notions useful to formulate the problem. The main contribution is stated in Section 3. Building on the coordinate transformations of [28, 20], we introduce in Section 4 a form for the closed-loop system which is convenient for the analysis discussed in Section 5). For the sake of simplicity, not all the proofs are presented, and they can be found in [5]. In the conclusions, it is emphasized how the proposed solution is also robust with respect to packet drop-out. The rest of the section summarizes the notation adopted in the paper.

Notation. Given an integer $1 \le i \le \nu$, the vector $(a_i, \ldots, a_\nu) \in \mathbb{R}^{\nu-i+1}$ will be succinctly denoted by the corresponding uppercase letter with index i, i.e. A_i. For $i = 1$, we will equivalently use the symbol A_1 or simply a. I_i denotes the $i \times i$ identity matrix. $\mathbf{0}_{i\times j}$ (respectively, $\mathbf{1}_{i\times j}$) denotes an $i \times j$ matrix whose entries are all 0 (respectively, 1). When only one index is present, it is meant that the matrix is a (row or column) vector.

If x is a vector, $|x|$ denotes the standard Euclidean norm, i.e. $|x| = \sqrt{x^T x}$, while $|x|_\infty$ denotes the infinity norm $\max_{1\le i\le n} |x_i|$. The vector $(x^T \; y^T)^T$ will be simply denoted as (x, y). $\mathbb{Z}_+$ (respectively, $\mathbb{R}_+$) is the set of nonnegative integers (real numbers), $\mathbb{R}^n_+$ is the positive orthant of $\mathbb{R}^n$. A matrix M is said to be Schur stable if all its eigenvalues are strictly inside the unit circle.

The symbol $\mathrm{sgn}(x)$, with x a scalar variable, denotes the sign function which is equal to 1 if $x > 0$, 0 if $x = 0$, and equal to -1 otherwise. If x is an n-dimensional vector, then $\mathrm{sgn}(x)$ is an n-dimensional vector whose ith component is given by $\mathrm{sgn}(x_i)$. Moreover, $\mathrm{diag}(x)$ is an $n \times n$ diagonal matrix whose element (i, i) is x_i.

Given a vector-valued function of time $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, the symbol $||x(\cdot)||_\infty$ denotes the supremum norm $||x(\cdot)||_\infty = \sup_{t\in\mathbb{R}_+} |x(t)|$. In the paper, two time scales are used, one denoted by the variable t in which the delay is θ, and the other one denoted by r, in which the delay is τ. Depending on the time scale, the following two norms are used: $||x_t|| = \sup_{-\theta\le\varsigma\le 0} |x(t+\varsigma)|$, $||x_r|| = \sup_{-\tau\le\sigma\le 0} |x(r+\sigma)|$. Moreover, $x(\bar{t}^+)$ represents the right limit $\lim_{t\to\bar{t}^+} x(t)$.

The saturation function [20] $\sigma : \mathbb{R} \to \mathbb{R}$ is an odd $\mathcal{C}^1$ function such that $0 \le \sigma'(s) \le 1$ for all $s \in \mathbb{R}$, $\sigma(s) = 1$ for all $s \ge 21/20$, and $\sigma(s) = s$ for all $0 \le s \le 19/20$. Furthermore, $\sigma_i(s) = \varepsilon_i \sigma(s/\varepsilon_i)$, with ε_i a positive real number.

2 Preliminaries

Consider a nonlinear system in feedforward form [29, 22, 12, 19], that is a system of the form

$$\dot{x}(t) = f(x(t), u(t)) := \begin{pmatrix} x_2(t) + h_1(X_2(t)) \\ \cdots \\ x_n(t) + h_{n-1}(X_n(t)) \\ u(t) \end{pmatrix}, \tag{1}$$

where $x_i(t) \in \mathbb{R}$, $X_i(t)$ is the vector of state variables $x_i(t), x_{i+1}(t), \ldots, x_n(t)$, $u(t) \in \mathbb{R}$, each function h_i is C^2, and there exists a positive real number $M > 0$ such that for all $i = 1, 2, \ldots, n-1$, if $|X_{i+1}|_\infty \le 1$, then

$$|h_i(X_{i+1})| \le M|X_{i+1}|^2 . \tag{2}$$

We additionally assume that a bound on the compact set of initial conditions is available to both the encoder and the decoder, namely a vector $\bar{\ell} \in \mathbb{R}^n_+$ is known for which

$$|x_i(t_0)| \le \bar{\ell}_i \,, \quad i = 1, 2, \ldots, n \,. \tag{3}$$

We investigate the problem of stabilizing the system above, when the measurements of the state variables travel through a communication channel. There are several ways to model the effect of the channel. In the present setting, we assume that there exists a sequence of strictly increasing transmission times $\{t_k\}_{k\in\mathbb{Z}_+}$, satisfying

$$T_m \le t_{k+1} - t_k \le T_M \,, \quad k \in \mathbb{Z}_+ \tag{4}$$

for some positive and known constants T_m, T_M, at which a packet of $N(t_k)$ bits, encoding the feedback information, is transmitted. The packet is received at the other end of the channel θ units of time later, namely at the times $\theta_k := t_k + \theta$. In problems of control under communication constraints, it is interesting to characterize how often the sensed information is transmitted to the actuators. In this contribution, as a measure of the data rate employed by the communication scheme we adopt the *average data rate* [27] defined as

$$R_{av} = \limsup\nolimits_{k\to+\infty} \sum_{j=0}^{k} \frac{N(t_j)}{t_k - t_0} \,, \tag{5}$$

where $\sum_{j=0}^{k} N(t_j)$ is the total number of bits transmitted during the time interval $[t_0, t_k]$. An *encoder* carries out the conversion of the state variable into packets of bits. At each time t_k, the encoder first samples the state vector to obtain $x(t_k)$, and then determines a vector $y(t_k)$ of symbols which can be transmitted through the channel. We recall below the encoder which has been proposed in [4], inspired by [27, 18], and then propose a modification to handle the presence of the delay. The encoder in [4] is as follows:

$$\begin{aligned}
\dot{\xi}(t) &= f(\xi(t), u(t)) & & \\
\dot{\ell}(t) &= \mathbf{0}_n & & t \ne t_k \\
\xi(t^+) &= \xi(t) + g_{\mathcal{E}}(x(t), \xi(t), \ell(t)) & & \\
\ell(t^+) &= \Lambda \ell(t) & & t = t_k \\
y(t^+) &= \mathrm{sgn}(x(t) - \xi(t)) & & t = t_k \,,
\end{aligned} \tag{6}$$

where ξ, ℓ is the encoder state, y is the feedback information transmitted through the channel, Λ is a Schur stable matrix, and $g_{\mathcal{E}}(x, \xi, \ell) = 4^{-1}\mathrm{diag}\,[\mathrm{sgn}\,(x - \xi)]\,\ell$. Note that each component of y takes value in $\{0, \pm 1\}$, therefore y can be transmitted as a packet of bits of finite length. In particular, if ξ_i is on the left of x_i then $+1$ is transmitted, if it is on the right, then -1 is transmitted. The system above is an *impulsive* system ([1, 14]) and its behavior is easily explained. At $t = t_0$, given an initial condition $\xi(t_0), \ell(t_0)$, the updates $\xi(t_0^+), \ell(t_0^+)$ of the encoder state and $y(t^+)$ of the output are obtained. The former update serves as initial condition for the continuous-time dynamics, and the state $\xi(t), \ell(t)$ is computed over the interval $[t_0, t_1]$. At the endpoint of the interval, a new update $\xi(t_1^+), \ell(t_1^+)$ is obtained and the procedure can be iterated an infinite number of times to compute the solution $\xi(t), \ell(t)$ for all t.

At the other end of the channel lies a decoder, which receives the packets $y(t_k)$, and reconstructs the state of the system. The decoder is very similar to the encoder. In fact, we have:

$$\begin{array}{rll} \dot{\psi}(t) &= f(\psi(t), u(t)) & \\ \dot{\nu}(t) &= \mathbf{0}_n & t \neq t_k \\ \psi(t^+) &= \psi(t) + g_{\mathcal{D}}(y(t), \nu(t)) & \\ \nu(t^+) &= \Lambda \nu(t) & t = t_k \end{array} \tag{7}$$

with $g_{\mathcal{D}}(y, \nu) = 4^{-1}\mathrm{diag}(y)\nu$. The control law is

$$u(t) = \alpha(\psi(t)) \ , \tag{8}$$

where α is the nested saturated function specified later. Note that this control law is feasible because the decoder and the actuator are *co-located*. If the encoder and the decoder agree to set their initial conditions to the same value, then it is not hard to see ([4]) that $\xi(t) = \psi(t)$ and $\ell(t) = \nu(t)$ for all t. Moreover, one additionally proves that $\xi(t)$ is an asymptotically correct estimate of $x(t)$, and the latter converges to zero [4].

When a delay affects the channel, the decoder does not know the first state sample throughout the interval $[t_0, t_0 + \theta]$, and hence it can not provide any feedback control action. The control is therefore set to zero. As the successive samples $y(t_k)$ are all received at times $\theta_k = t_k + \theta$, the decoder becomes aware of the value of ξ θ units of time later. Hence, the best one can expect is to reconstruct the value of $\xi(t - \theta)$ (see Lemma 1 below), and to this purpose the following decoder is proposed:

$$\begin{array}{rll} \dot{\psi}(t) &= f(\psi(t), \alpha(\psi(t-\theta))) & \\ \dot{\nu}(t) &= \mathbf{0}_n & t \neq \theta_k \\ \psi(t^+) &= \psi(t) + g_{\mathcal{D}}(y(t-\theta), \nu(t)) & \\ \nu(t^+) &= \Lambda \nu(t) & t = \theta_k \\ u(t) &= \alpha(\psi(t)) \ . & \end{array} \tag{9}$$

We also need to modify the encoder. Indeed, as mentioned in the case with no delay, for the encoder to work correctly, the control law (8), and hence $\psi(t)$, must be available to the encoder. To reconstruct this quantity, the following equations are added to the encoder (6):

$$\begin{array}{rll} \dot{\omega}(t) &= f(\omega(t), \alpha(\omega(t-\theta))) & t \neq \theta_k \\ \omega(t^+) &= \omega(t) + g_{\mathcal{E}}(x(t-\theta), \xi(t-\theta), \ell(t-\theta)) & t = \theta_k \ . \end{array}$$

As in [28, 20], we shall adopt a *linear* change of coordinates in which the control system takes a special form convenient for the analysis. Differently from [4], this change of coordinates plays a role also in the encoding/decoding procedure. Indeed, denoted by Φ the nonsingular matrix which defines the change of coordinates, and which we define in detail in Section 4, the functions $g_{\mathcal{E}}$, $g_{\mathcal{D}}$ which appear in (11) and, respectively, (9) are modified as

$$g_{\mathcal{E}}(x,\xi,\ell) = (4\Phi)^{-1}\mathrm{diag}\left[\mathrm{sgn}\left(\Phi(x-\xi)\right)\right]\ell\ , \quad g_{\mathcal{D}}(y,\nu) = (4\Phi)^{-1}\mathrm{diag}(y)\nu\ ,$$

the initial conditions of the encoder and decoder are set as

$$\begin{array}{c} ||\omega_{\theta_0}|| = 0\ , \quad \xi(t_0) = \mathbf{0}_n\ , \quad \ell(t_0) = 2\Phi\bar{\ell}\ , \\ ||\psi_{\theta_0}|| = 0\ , \quad \nu(\theta_0) = 2\Phi\bar{\ell}\ , \end{array} \tag{10}$$

and, finally, the vector y which is transmitted through the channel take the expression

$$y(t^+) = \mathrm{sgn}(\Phi(x(t) - \xi(t)))\ .$$

Overall, the equations which describe the encoder are:

$$\begin{array}{rll} \dot{\omega}(t) & = f(\omega(t), \alpha(\omega(t-\theta))) & t \neq \theta_k \\ \dot{\xi}(t) & = f(\xi(t), \alpha(\omega(t))) & \\ \dot{\ell}(t) & = \mathbf{0}_n & t \neq t_k \\ \omega(t^+) & = \omega(t) + g_{\mathcal{E}}(x(t-\theta), \xi(t-\theta), \ell(t-\theta)) & t = \theta_k \\ \xi(t^+) & = \xi(t) + g_{\mathcal{E}}(x(t), \xi(t), \ell(t)) & \\ \ell(t^+) & = \Lambda\ell(t) & \\ y(t^+) & = \mathrm{sgn}(\Phi(x(t) - \xi(t))) & t = t_k\ . \end{array} \tag{11}$$

The following can be easily proven.

Lemma 1. *In the above setting, we have: (i) $\omega(t) = \psi(t)$ for all $t \geq t_0$, (ii) $\xi(t-\theta) = \psi(t)$ and $\nu(t-\theta) = \ell(t)$ for all $t \geq \theta_0$.*

As anticipated, the encoder and decoder we introduced above are such that the internal state of the former is exactly reconstructed from the internal state of the latter. This also implies that in the analysis to come it is enough to focus on the equations describing the process and the decoder only.

3 Main Result

The problem we tackle in this paper is, given any value of the delay θ, find the matrices Λ, Φ in (11) and (9), and the control (8) which guarantee the state of the entire closed-loop system to converge to the origin. As recalled in the previous section, at times t_k, the measured state is sampled, packed into a sequence of $N(t_k)$ bits, and fed back to the controller. In other words, the information flows from the sensors to the actuators with an average rate

R_{av} given by (5). In this setting, it is therefore meaningful to formulate the problem of stabilizing the system *while* transmitting the minimal amount of feedback information per unit of time, that is using the minimal average data rate. The problem can be formally cast as follows.

Definition 1. *System (1) is semi-globally asymptotically and locally exponentially stabilizable using an average data rate arbitrarily close to the infimal one if, for any* $\bar{\ell} \in \mathbb{R}_+^n$, $\theta > 0$, $\hat{R} > 0$, *an encoder (11), a decoder (9), initial conditions (3), (10), and a controller (8) exist such that for the closed-loop system with state* $X := (x, \omega, \xi, \ell, \psi, \nu)$, *we have the following properties.*

(i) The origin is a stable equilibrium point;
(ii) There exist a compact set C *containing the origin, and* $T > 0$, *such that* $X(t) \in C$ *for all* $t \geq T$*;*
(iii) For all $t \geq T$, *for some positive real numbers* k, δ,

$$|X(t)| \leq k||X_T|| \exp(-\delta(t-T)) \; .$$

(iv) $R_{av} < \hat{R}$.

Remark 1. It is straightforward to verify that the origin is indeed an equilibrium point for the closed-loop system. Moreover, item (iii) explains what is meant by stabilizability using an average data rate arbitrarily close to the infimal one. As a matter of fact, (iv) requires that the average data rate can be made arbitrarily close to the zero, which of course is the infimal data rate. It is “infimal” rather than “minimal”, because we could never stabilize an open-loop unstable system such as (1) with a zero data rate (no feedback). ◁

Compared with the papers [29, 22, 12, 19], concerned with the stabilization problem of nonlinear feedforward systems, the novelty here is due to the presence of impulses, quantization noise which affects the measurements and delays which affect the control action (on the other hand, we neglect parametric uncertainty, considered in [19]). In [30], it was shown robustness with respect to measurement errors for non-impulsive systems with no delay. In [20], the input is delayed, but neither impulses nor measurement errors are present. Impulses and measurement errors are considered in [4], where the minimal data rate stabilization problem is solved, but instantaneous delivery of the packets is assumed.
We state the main result of the paper.

Theorem 1. *System (1) is semi-globally asymptotically and locally exponentially stable with an average data rate arbitrarily close to the infimal one.*

Remark 2. The proof is constructive and explicit expressions for Λ, Φ, and the controller are determined. ◁

Remark 3. This result can be viewed as a nonlinear generalization of the well-known data rate theorem for linear systems. Indeed, the linearization of the feedforward system at the origin is a chain of integrators, for which the minimal data rate theorem for linear systems states that stabilizability is possible using an average data rate arbitrarily close to zero. ◁

4 Change of Coordinates

Building on the coordinate transformations in [20, 28], we put the system composed of the process and the decoder in a special form. Before doing this, we recall that for feedforward systems encoders, decoders and controllers are designed in a recursive way [28, 29, 22, 12, 20, 4]. In particular, at each step $i = 1, 2, \ldots, n$, one focuses on the last $n - i + 1$ equations of system (1), design the last $n - i + 1$ equations of the encoder and the decoder, the first i terms of the nested saturated controller, and then proceed to the next step, where the last $n-i$ equations of (1) are considered. To this end, it is useful to introduce additional notation to denote these subsystems. In particular, for $i = 1, 2, \ldots, n$, we denote the last $n - i + 1$ equations of (1) by

$$\dot{X}_i(t) = H_i(X_{i+1}(t), u(t)) = \begin{pmatrix} x_{i+1}(t) + h_i(X_{i+1}(t)) \\ \cdots \\ x_n(t) + h_{n-1}(X_n(t)) \\ u(t) \end{pmatrix} , \tag{12}$$

with $u(t) = \alpha(\psi(t))$, while for the last $n - i + 1$ equations of the decoder (9) we adopt the notation

$$\begin{aligned} &\dot{\Psi}_i(t) = H_i(\Psi_{i+1}(t), u(t-\theta)) \\ &\dot{N}_i(t) = \mathbf{0}_{n-i+1} && t \neq \theta_k \ , \\ &\Psi_i(t) = \Psi_i(t^-) + (4\Phi_i)^{-1}\mathrm{diag}(Y_i(t-\theta))N_i(t^-) \\ &N_i(t) = \Lambda_i N_i(t^-) && t = \theta_k \ , \end{aligned} \tag{13}$$

where N_i denotes the components from i to n of ν. Moreover, for given positive constants $L \leq M$, $\kappa \geq 1$, with M defined in (2), we define the *non singular positive* matrices[1] Φ_i as:

$$\Phi_i X_i := \begin{bmatrix} p_i\left(\dfrac{M}{L}\kappa^{i-1}x_i, \ldots, \dfrac{M}{L}\kappa^{n-1}x_n\right) \\ \cdots \\ p_n\left(\dfrac{M}{L}\kappa^{n-1}x_n\right) \end{bmatrix} , \quad i = 1, \ldots, n \ , \tag{14}$$

where the functions $p_i, q_i : \mathbb{R}^{n-i+1} \to \mathbb{R}$ are [28, 20]

$$\begin{aligned} p_i(a_i, \ldots, a_n) &= \sum_{j=i}^{n} \frac{(n-i)!a_j}{(n-j)!(j-i)!} \ , \\ q_i(a_i, \ldots, a_n) &= \sum_{j=i}^{n} \frac{(-1)^{i+j}(n-i)!a_j}{(n-j)!(j-i)!} \ , \end{aligned}$$

[1] The matrix Φ_1 will be simply referred to as Φ.

with $p_i(q_i(a_i, \ldots, a_n), \ldots, q_n(a_n)) = a_i$, $q_i(p_i(a_i, \ldots, a_n), \ldots, p_n(a_n)) = a_i$. Finally, let us also introduce the change of *time scale*

$$t = \kappa r \ , \tag{15}$$

and the *input* coordinate change

$$v(r) = \kappa p_n \left(\frac{M}{L} \kappa^{n-1} u(\kappa r) \right) \ . \tag{16}$$

Then we have the following.

Lemma 2. *Let* $i \in \{1, 2, \ldots, n\}$ *and*

$$\tau := \theta/\kappa \ , \quad r_k := t_k/\kappa \ , \quad \rho_k := \theta_k/\kappa \ . \tag{17}$$

The change of coordinates (15), (16), and

$$\begin{array}{l} Z_i(r) = \Phi_i X_i(\kappa r) \\ E_i(r) = \Phi_i(\Psi_i(\kappa r) - X_i(\kappa(r - \tau))) \ , \\ P_i(r) = N_i(\kappa r) \end{array} \tag{18}$$

transforms system (12)–(13) into

$$\begin{array}{ll} \dot{Z}_i(r) = \Gamma_i Z_i(r) + \mathbf{1}_{n-i+1} v(r) + \varphi_i(Z_{i+1}(r)) & \\ \dot{E}_i(r) = \Gamma_i E_i(r) + \varphi_i(E_{i+1}(r) + Z_{i+1}(r - \tau)) & \\ \qquad -\varphi_i(Z_{i+1}(r - \tau)) & \\ \dot{P}_i(r) = \mathbf{0}_{n-i+1} & r \neq \rho_k \\ Z_i(r^+) = Z_i(r) & \\ E_i(r^+) = E_i(r) + 4^{-1} \mathrm{diag}(\mathrm{sgn}(-E_i(r))) P_i(r) & \\ P_i(r^+) = \Lambda_i P_i(r) & r = \rho_k \ , \end{array} \tag{19}$$

where

$$\Gamma_i := \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \ , \quad \varphi_i(Z_{i+1}) := \begin{bmatrix} f_i(Z_{i+1}) \\ f_{i+1}(Z_{i+2}) \\ \ldots \\ f_{n-1}(Z_n) \\ 0 \end{bmatrix} \ .$$

Proof. It is shown in [20] that, (15), (16) and $Z_i(r) = \Phi_i X_i(\kappa r)$ transforms (12) into

$$\begin{aligned} \dot{Z}_i(r) \ &= \ F_i(Z_{i+1}(r), v(r)) \\ &:= \begin{bmatrix} \sum_{j=i+1}^{n} z_j(r) + v(r) + f_i(Z_{i+1}(r)) \\ \sum_{j=i+2}^{n} z_j(r) + v(r) + f_{i+1}(Z_{i+2}(r)) \\ \vdots \\ v(r) \end{bmatrix} \ , \end{aligned} \tag{20}$$

where

$$|f_i(Z_{i+1})| \le P|Z_{i+1}|^2 , \quad P = n^3(n!)^3 L\kappa^{-1} , \tag{21}$$

provided that $|Z_{i+1}|_\infty \le (M\kappa)/(L(n+1)!)$. Clearly, the equation (20) is equal to the first equation of (19). Bearing in mind (20), and by differentiating E_i defined in (18), we obtain

$$\begin{aligned}
\dot{E}_i(r) &= F_i(E_{i+1}(r) + Z_{i+1}(r-\tau), v(r-\tau)) - F_i(Z_{i+1}(r-\tau), v(r-\tau)) \\
&= \Gamma_i(E_i(r) + Z_i(r-\tau)) + \mathbf{1}_{n-i+1}v(r-\tau) + \varphi_i(E_{i+1}(r) + Z_{i+1}(r-\tau)) \\
&\quad -\Gamma_i(Z_i(r-\tau)) - \mathbf{1}_{n-i+1}v(r-\tau) - \varphi_i(Z_{i+1}(r-\tau)) \\
&= \Gamma_i E_i(r) + \varphi_i(E_{i+1}(r) + Z_{i+1}(r-\tau)) - \varphi_i(Z_{i+1}(r-\tau))
\end{aligned} \tag{22}$$

for $r \ne \rho_k$, while for $r = \rho_k$, we have:

$$\begin{aligned}
E_i(\rho_k^+) &= \Phi_i(\Psi_i(\kappa\rho_k^+) - X_i(\kappa(\rho_k - \tau)^+)) \\
&= \Phi_i(\Psi_i(\theta_k^+) - X_i(t_k^+)) \\
&= \Phi_i(\Psi_i(\theta_k) + (4\Phi_i)^{-1}\mathrm{diag}(Y_i(t_k^+))N_i(\theta_k) - X_i(t_k)) \\
&= \Phi_i(\Psi_i(\theta_k) - X_i(t_k)) + 4^{-1}\mathrm{diag}(\mathrm{sgn}(\Phi_i[X_i(t_k) - \Xi_i(t_k)]))N_i(\theta_k) \\
&= \Phi_i(\Psi_i(\theta_k) - X_i(t_k)) + 4^{-1}\mathrm{diag}(\mathrm{sgn}(\Phi_i[X_i(t_k) - \Psi_i(\theta_k)]))N_i(\theta_k) ,
\end{aligned} \tag{23}$$

where the last equality descends from (ii) in Lemma 1, and implies

$$E_i(\rho_k^+) = E_i(\rho_k) + 4^{-1}\mathrm{diag}(\mathrm{sgn}(-E_i(\rho_k)))N_i(\theta_k) . \tag{24}$$

The thesis then follows if we observe that the variable P_i defined in (18) satisfies

$$\begin{aligned}
\dot{P}_i(r) &= \mathbf{0}_{n-i+1} \;\; r \ne \rho_k \\
P_i(r^+) &= \Lambda_i P_i(r) \;\; r = \rho_k .
\end{aligned} \tag{25}$$

□

Before ending the section, we specify the nested saturated controller $u(t) = \alpha(\psi(t))$ which is shown to stabilize the closed-loop system in the next section. In particular, we have

$$\begin{aligned}
\alpha(\psi(t)) = -\frac{L}{M\kappa^n}\sigma_n\Bigg(p_n\left(\kappa^{n-1}\frac{M}{L}\psi_n(t)\right) + \sigma_{n-1}\Bigg(p_{n-1}\Bigg(\kappa^{n-2}\frac{M}{L}\psi_{n-1}(t), \\
\kappa^{n-1}\frac{M}{L}\psi_n(t)\Bigg) + \ldots + \sigma_i\left(p_i\left(\kappa^{i-1}\frac{M}{L}\psi_i(t), \ldots, \kappa^{n-1}\frac{M}{L}\psi_n(t)\right)\right. \\
+\lambda_{i-1}(t))\ldots\Bigg) ,
\end{aligned}$$

with

$$\begin{aligned}
\lambda_{i-1}(t) = \sigma_{i-1}\Bigg(p_{i-1}\left(\kappa^{i-2}\frac{M}{L}\psi_{i-1}(t), \ldots, \kappa^{n-1}\frac{M}{L}\psi_n(t)\right) \\
+ \ldots + \sigma_1\left(p_1\left(\frac{M}{L}\psi_1(t), \ldots, \kappa^{n-1}\frac{M}{L}\psi_n(t)\right)\right)\ldots\Bigg) ,
\end{aligned}$$

and where the saturation levels ε_i of $\sigma_i(r) = \varepsilon_i\sigma(r/\varepsilon_i)$ are chosen as follows:

$$1 = 80\varepsilon_n = 80^2\varepsilon_{n-1} = \ldots = 80^n\varepsilon_1 \; . \tag{26}$$

In the new coordinates (15)–(16), (18), the controller takes the form

$$\begin{aligned} v(r) = &-\sigma_n(e_n(r) + z_n(r-\tau) + \sigma_{n-1}(e_{n-1}(r) + z_{n-1}(r-\tau) + \\ &\ldots + \sigma_i(e_i(r) + z_i(r-\tau) + \hat{\lambda}_{i-1}(r))\ldots)) \; , \end{aligned} \tag{27}$$

with $\hat{\lambda}_{i-1}(r) = \sigma_{i-1}(e_{i-1}(r) + z_{i-1}(r-\tau) + \ldots + \sigma_1(e_1(r) + z_1(r-\tau))\ldots)$.

5 Analysis

In the previous sections, we have introduced the encoder, the decoder and the controller. In this section, in order to show the stability property, we carry out a step-by-step analysis, where at each step i, we consider the subsystem (19) in closed-loop with (27). We first introduce two lemmas which are at the basis of the iterative construction. The first one, which, in a different form, was basically given in [4], shows that the decoder asymptotically tracks the state of the process under a boundedness assumption. The proof we present here is more straightforward than the original one.

Lemma 3. *Suppose (3) is true. If for some $i = 1, 2, \ldots, n$ there exists a positive real number $\bar{\mathrm{Z}}_{i+1}$ such that* [2]

$$||Z_{i+1}(\,\cdot\,)||_\infty \le \bar{\mathrm{Z}}_{i+1} \; ,$$

and, for all $r \ge \rho_0$,

$$|e_j(r)| \le p_j(r)/2 \; , \quad j = i+1, i+2, \ldots, n \; ,$$

with[3]

$$P_{i+1}(\rho^+) = \Lambda_{i+1}P_{i+1}(\rho) \quad \rho = \rho_k \; ,$$

and Λ_{i+1} a Schur stable matrix, then for all $r \ge \rho_0$,

$$|e_i(r)| \le p_i(r)/2 \; ,$$

with $p_i(r^+) = p_i(r)/2$, for $r = \rho_k$, if $i = n$, and

$$\begin{bmatrix} p_i(r^+) \\ P_{i+1}(r^+) \end{bmatrix} = \begin{bmatrix} 1/2 & * \\ \mathbf{0}_{n-i} & \Lambda_{i+1} \end{bmatrix} \begin{bmatrix} p_i(r) \\ P_{i+1}(r) \end{bmatrix} r = \rho_k \; , \tag{28}$$

if $i \in \{1, 2, \ldots, n-1\}$, where $*$ is a $1 \times (n-i)$ row vector depending on $\bar{Z}_{i+1}$, $\bar{\ell}$, and T_M.

---

[2] The conditions are void for $i = n$.

[3] In the statement, the continuous dynamics of the impulsive systems are trivial – the associated vector fields are identically zero – and hence omitted.

Proof. Recall first (22), (24). Furthermore, by (10), the definition of Φ, and (3), $|e_j(\rho_0)| \le p_j(\rho_0)/2$ for $j = i, i+1, \ldots, n$. For $i = n$, as $|e_n(\rho_0)| \le p_n(\rho_0)/2$, it is immediately seen that

$$|e_n(\rho_0^+)| = |e_n(\rho_0) + 4^{-1}\mathrm{sgn}(-e_n(\rho_0))p_n(\rho_0)| \le 4^{-1}p_n(\rho_0)$$

which proves that $|e_n(\rho_0^+)| \le p_n(\rho_0^+)/2$, provided that $\Lambda_n = 1/2$. As $\dot{e}_n(r) = 0$, then $|e_n(r)| \le p_n(\rho_0^+)/2$ for $r \in [\rho_0, \rho_1)$. As $\dot{p}_n(r) = 0$, also $|e_n(\rho_1)| \le p_n(\rho_1)/2$, and iterative arguments prove that $|e_n(r)| \le p_n(\rho_k^+)/2$ on each interval $[\rho_k, \rho_{k+1})$. Notice that the single trivial eigenvalue of Λ_n is strictly less than the unity. The first equation of (22) writes as:

$$\begin{aligned}\dot{e}_i(r) &= \mathbf{1}_{n-i}E_{i+1}(r) + \varphi_i(E_{i+1}(r) + Z_{i+1}(r-\tau)) - \varphi_i(Z_{i+1}(r-\tau)) \\ &= \left(\mathbf{1}_{n-i} + \left[\frac{\partial \varphi_i(y_{i+1})}{\partial y_{i+1}}\right]_{\alpha(r)E_{i+1}(r)+Z_{i+1}(r-\tau)}\right) E_{i+1}(r)\ ,\end{aligned}$$

with $\alpha(r) \in [0,1]$ for all r. As both E_{i+1} and Z_{i+1} are bounded, it is not hard to see [4] that there exists a positive real number F_i depending on $\bar{\mathrm{Z}}_{i+1}$ and $\bar{\ell}$, such that, for $r \in [\rho_k, \rho_{k+1})$,

$$e_i(r) \le e_i(\rho_k^+) + F_i(\rho_{k+1} - \rho_k) \sum_{j=i+1}^{n} p_j(\rho_k^+)/2\ ,$$

with $|e_i(\rho_0^+)| \le p_i(\rho_0)/4$. By iteration, the thesis is inferred provided that

$$\begin{aligned}p_i(\rho_k^+) &= \frac{1}{2}p_i(\rho_k) + F_i T_M \mathbf{1}_{n-i}\Lambda_{i+1}P_{i+1}(\rho_k) \\ &\ge \frac{1}{2}p_i(\rho_k) + F_i(\rho_{k+1} - \rho_k) \sum_{j=i+1}^{n} p_j(\rho_k^+)\ .\end{aligned}$$

Note that, by the definition of $p_i(\rho_k^+)$ above, $P_i(\rho_k^+) = \Lambda_i P_i(\rho_k)$, with Λ_i the matrix in (28), that shows Λ_i to be a Schur stable matrix provided that so is Λ_{i+1}. □

The following remark will be useful later on.

Remark 4. From the proof of the lemma, it is possible to see that, if $||z(\cdot)||_\infty \le Z$, for some $Z > 0$, then e and p in (19) (with $i = 1$) obey the equations[4]

$$\begin{array}{ll}\dot{e}(r) = A(r)e(r) & \\ \dot{p}(r) = \mathbf{0}_n & r \ne \rho_k \\ e(r) = e(r^-) + 4^{-1}\mathrm{diag}[\mathrm{sgn}(-e(r^-))]p(r^-) & \\ p(r) = \Lambda p(r^-) & r = \rho_k\ ,\end{array} \tag{29}$$

[4] Again, we adopt the symbol Λ rather than Λ_1.

with

$$A(r) := \begin{bmatrix} 0 & a_{12}(r) & a_{13}(r) & \dots & a_{1\,n-1}(r) & a_{1n}(r) \\ 0 & 0 & a_{23}(r) & \dots & a_{2\,n-1}(r) & a_{2n}(r) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{n-1\,n}(r) \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{30}$$

and where the off-diagonal components of A, rather than as functions of $(r, e(r), z(r-\tau))$, are viewed as bounded (unknown) functions of r, whose absolute value can be assumed without loss of generality to be upper bounded by a positive constant depending on Z, $\bar{\ell}$ and T_M. $\triangleleft$

The next statement, based on Lemma 10 in [20], shows that a controller exists which guarantees the boundedness of the state variables, a property required in the latter result. Note that the arguments of the proof in [20] hold even in the presence of a "measurement" disturbance e induced by the quantization, which can be possibly large during the transient but it is decaying to zero asymptotically.

Lemma 4. *Consider the system*

$$\dot{Z}(r) = -\varepsilon\sigma\left[\frac{1}{\varepsilon}(Z(r-\tau) + e(r) + \lambda(r))\right] + \mu(r)$$

where $Z \in \mathbb{R}$, ε is a positive real number, and additionally:

- *$\lambda(\,\cdot\,)$ and $\mu(\,\cdot\,)$ are continuous functions for which positive real numbers λ_* and μ_* exist such that, respectively, $|\lambda(r)| \le \lambda_*$, $|\mu(r)| \le \mu_*$, for all $r \ge r_0$;*
- *$e(\,\cdot\,)$ is a piecewise-continuous function for which a positive time r_* and a positive number e_* exist such that $|e(r)| \le e_*$, for all $r \ge r_*$.*

If

$$\tau \in \left(0, \frac{1}{24}\right], \quad \lambda_* \in \left(0, \frac{\varepsilon}{80}\right], e_* \in \left(0, \frac{\varepsilon}{80}\right], \quad \mu_* \in \left(0, \frac{\varepsilon}{80}\right],$$

then there exist positive real numbers Z_* and $R \ge 0$ such that $||Z(\,\cdot\,)||_\infty \le Z_*$, and for all $r \ge R$,

$$|Z(r)| \le 4(\lambda_* + \mu_* + e_*) .$$

Remark 5. The upper bounds on λ_*, e_*, μ_* could be lowered to $\varepsilon/40$ and the result would still hold. The more conservative bounds are needed in forthcoming applications of the lemma. $\triangleleft$

To illustrate the iterative analysis in a concise manner, the following is very useful (cf. [20]).

Inductive Hypothesis There exists $\bar{Z}_i > 0$ such that $||Z_i(\,\cdot\,)|| \le \bar{Z}_i$. Moreover, for each $j = i, i+1, \dots, n$, $|e_j(r)| \le p_j(r)/2$, for all $r \ge \rho_0$, and there exists $R_i > \tau$ such that for all $r \ge R_i$,

$$|z_j(r)| \le \frac{1}{4}\varepsilon_j\ , \quad |e_j(r)| \le \frac{1}{2n}\cdot\frac{1}{80^{j-i+2}}\varepsilon_j\ .$$

Initial step $(i = n)$ The initial step is trivially true, provided that $\tau \le 1/24$, and $\varepsilon_n = 1/80$. Indeed, consider the closed-loop system (19), (27) with $i = n$, to obtain:

$$\begin{array}{ll} \begin{aligned} \dot z_n(r) &= -\sigma_n(z_n(r-\tau) + e_n(r) + \hat\lambda_{n-1}(r)) \\ \dot e_n(r) &= 0 \\ \dot p_n(r) &= 0 \end{aligned} & r \ne \rho_k \\ \begin{aligned} z_n(r^+) &= z_n(r) \\ e_n(r^+) &= e_n(r) + 4^{-1}\mathrm{sgn}(-e_n(r))p_n(r) \\ p_n(r^+) &= \Lambda_n p_n(r) \end{aligned} & r = \rho_k\ , \end{array} \tag{31}$$

where we set $\Lambda_n := 1/2$. By Lemma 3 and (31), $|e_n(r)| \le \varepsilon_n/80$ from a certain time R_n' on. Applying Lemma 4 to the z_n sub-system, we conclude that $||z_n(\,\cdot\,)||_\infty \le \bar{Z}_n$, and there exists a time $R_n > R_n'$ such that $|z_n(r)| \le \varepsilon_n/4$, and $|e_n(r)| \le \varepsilon_{n-1}/(n\cdot 160)$ for all $r \ge R_n$, the latter again by Lemma 3.

Inductive step The inductive step is summarized in the following result.

Lemma 5. *Let*

$$P \le P_m \le [20\cdot(80)^n n]^{-1}\ , \quad \tau \le \tau_m \le [4\cdot 80^{n+1}n(n+2)]^{-1}\ . \tag{32}$$

If the inductive hypothesis is true for some $i \in \{2,\ldots,n\}$, then it is also true for $i-1$.

Applying this lemma repeatedly, one concludes that, after a finite time, the state converge to the linear operation region for all the saturation functions, and the closed-loop system starts evolving according to the equations (cf. Remark 4)

$$\begin{array}{ll} \begin{aligned} \dot z(r) &= A_1 z(r) + A_2 z(r-\tau) + A_2 e(r) + \varphi(z(r)) \\ \dot e(r) &= A(r)e(r) \\ \dot p(r) &= \mathbf{0}_n \end{aligned} & r \ne \rho_k \\ \begin{aligned} z(r^+) &= z(r) \\ e(r^+) &= e(r) + 4^{-1}\mathrm{diag}[\mathrm{sgn}(-e(r))]p(r) \\ p(r^+) &= \Lambda p(r) \end{aligned} & r = \rho_k\ , \end{array} \tag{33}$$

where:

(i) A_1, A_2 are matrices for which there exist $q = (1+n^2)^{n-1}$, $a = n$, and $Q = Q^T > 0$ such that

$$(A_1 + A_2)^T Q + Q(A_1 + A_2) \le -I\ ,$$

with $||Q|| \le q$ and $||A_1||, ||A_2|| \le a$;

(ii) there exists $\gamma > 0$ such that $\varphi(z(r)) := [\, f_1(Z_2(r)) \; \ldots \; f_{n-1}(Z_n(r)) \; 0]^T$ satisfies

$$|\varphi(z)| \leq \gamma |z| \; ;$$

(iii) $A(r)$ is as in (30);

(iv) Λ is the Schur stable matrix designed following the proof of Lemma 3.

Remark 6. It can be shown that the same arguments used for the proofs of the Lemma 3 to 5 lead to the conclusion that there always exists a sufficiently small neighborhood of initial conditions for the system (19), (27), with $i = 1$, such that the entire state evolves in a set where all the saturation functions operate in their linear region. This remark is important to conclude Lyapunov stability of the closed-loop system. ◁

In [20] the authors investigate the stability property of

$$\dot{z}(r) = A_1 z(r) + A_2 z(r - \tau) + \varphi(z(r)) \; ,$$

that is the first component of system (33), with $e = 0$ and no impulses. In the present case, e is due to the quantization noise and drives the z-subsystem. The "driver" subsystem is composed of the (e, p) equations of (33). Hence, we have to study the stability of a *cascade* system with impulses. To this end, concisely rewrite the (e, p) equations of the system above as ([1])

$$\begin{aligned} \dot{\epsilon}(r) &= B(r)\epsilon(r) \;\; r \neq \rho_k \\ \epsilon(r^+) &= g_k(\epsilon(r)) \;\; r = \rho_k \; , \end{aligned} \tag{34}$$

with $\epsilon = (e, p)$, $|g_k(\epsilon)| \geq |\epsilon|/2$, and notice the following consequence of Lemma 3 above, and [1], Theorem 15.2.

Corollary 1. *There exists a function $V(r, \epsilon) = V(r, e, p) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that, for all $r \in \mathbb{R}_+$ and for all $\epsilon = (e, p) \in \mathbb{R}^n \times \mathbb{R}^n$ for which $|e| \leq |p|/2$, satisfies*

$$\begin{aligned} c_1 |\epsilon|^2 \leq V(r, \epsilon) &\leq c_2 |\epsilon|^2 \\ \frac{\partial V}{\partial r} + \frac{\partial V}{\partial \epsilon} B(r)\epsilon(r) &\leq -c_3 |\epsilon|^2 \;\; r \neq \rho_k \\ V(r^+, g_k(\epsilon)) &\leq V(r, \epsilon) \;\; r = \rho_k \\ \left| \frac{\partial V(r, \epsilon)}{\partial \epsilon} \right| &\leq c_4 |\epsilon| \; , \end{aligned}$$

for some positive constants c_i, $i = 1, \ldots, 4$.

The corollary points out that there exists an exponential Lyapunov function for the system (34). Based on this function, one can build a Lyapunov-Krasowskii functional to show that the origin is exponentially stable for the entire cascade impulsive system (33), thus extending Lemma 11 in [20] in the following way.

Lemma 6. *Consider system (33), for which the conditions (i)–(iv) hold. If*

$$\gamma \le \frac{1}{8q} \quad \textit{and} \quad \tau \le \min\left\{\frac{1}{16a^2(8aq+1)^2}, \frac{1}{32q^2a^4}, 2\right\} ,$$

then, for all $r \ge \rho_0$, for some positive real numbers k, δ, we have

$$|(z(r), \epsilon(r))| \le k||(z,\epsilon)_{\rho_0}|| \exp(-\delta(r-\rho_0)) .$$

We can now state the following stability result for the system (19), (27).

Proposition 1. *Consider the closed-loop system (19), (27) and let $|e_i(\rho_0)| \le p_i(\rho_0)/2$ for all $i = 1, 2, \ldots, n$. If (26) holds,*

$$L \le \min\{M, \frac{M\kappa}{(n+1)!}\} \tag{35}$$

and

$$\begin{aligned} 0 \le \tau \le \tau_m = &[\max\left\{4\cdot(80)^{n+1}n(n+2), 16n^2(8n(1+n^2)^{n-1}+1)^2,\right.\\ &\left.32(1+n^2)^{2(n-1)}n^4\right\}]^{-1} \qquad (36)\\ 0 \le P \le P_m = &[\max\{20\cdot 80^{n+1}n, 8(1+n^2)^{n-1}\sqrt{n(n-1)}\}]^{-1} , \end{aligned}$$

then the following properties hold.

(i) The origin of the closed-loop system is stable;
(ii) There exist a compact neighborhood $\hat{C}$ of the origin and $R > 0$ such that, for all $r \ge R$, the state belongs to $\hat{C}$;
(ii) For all $r \ge R$, for some positive real numbers $\hat{k}, \hat{\delta}$,

$$|(z(r), e(r), p(r))| \le \hat{k}||(z,e,p)_R|| \exp(-\hat{\delta}(r-R)) . \tag{37}$$

Proof. Bearing in mind (21) and that $\varepsilon_i < 1$, for $i = 1, 2, \ldots, n$, and $(M\kappa)/(L(n+1)!) \ge 1$, then γ in (ii) after (33) is seen to be equal to $\sqrt{n(n-1)}P$, and the condition $P \le [8(1+n^2)^{n-1}\sqrt{n(n-1)}]^{-1}$ in (36) actually implies $\gamma \le 1/(8q)$. Analogously, one can check that the requirements on τ and P in (36) imply that all the conditions in Lemma 5 and 6 are true. These lemma (see also Remark 6) allow us to infer the thesis. □

The proof of the main result of the paper simply amounts to rephrase the proposition above in the original coordinates. This is straightforward and we omit it. We only discuss briefly the issue of the minimality of the data rate. By definition of R_{av}, it is always possible to guarantee that $R_{av} < \hat{R}$, provided that $T_m \ge 2n/\hat{R}$. Now the stability results we presented hold for a given value of T_m which may or may not fulfill the inequality above. Suppose it does not. Can we increase T_m above $2n/\hat{R}$ and still have stability? The answer is yes, for the value of T_m (and hence of $T_M \ge T_m$) affects the entries of $A(r)$ and Λ, but the exponential stability of the (e,p) equations (and therefore of system

(29)) remains true, as it is evident from the proof of Lemma 3. Hence, the arguments above still apply and minimality of the data rate is proven.

We stress that the proof of the result is constructive, that is we give the explicit expressions of the encoder, the decoder and the controller which solve the problem. As a matter of fact, the equations of encoder and the decoder are introduced in (11) and, respectively, (9). The matrices Λ and Φ appearing there are, respectively, designed in Lemma 3, and defined in (14). The parameters of the nested saturated controller are L, κ and the saturation levels ε_i. The latter are defined in (26). The former must be chosen in such a way that (35) and (36) are satisfied. Bearing in mind the definitions (17), (21), it is easy to see that, for any value of the delay θ, there exist a sufficiently large value of κ and a sufficiently small value of L such that (35) and (36) are true. These values will be, respectively, larger and smaller than the corresponding values given in [20], as the presence of the quantization error requires a stronger control action.

6 Conclusion

We have shown that minimal data rate stabilization of nonlinear systems is possible even when the communication channel is affected by an arbitrarily large transmission delay. The system has been modeled as the feedback interconnection of a couple of impulsive nonlinear control systems with the delay affecting the feedback loop. In suitable coordinates, the closed-loop system turns out to be described by a cascade of impulsive delay nonlinear control systems, and semi-global asymptotic plus local exponential stability has been shown. The proof relies, among other things, on the design of a Lyapunov-Krasowskii functional for an appropriate cascade impulsive time-delay system. If the encoder is endowed with a device able to detect abrupt changes in the rate of growth of x_n, or if a dedicated channel is available to inform the encoder about the transmission delays, then it is not difficult to derive the same kind of stability result for the case when the delays are time-varying and upper-bounded by θ. Similarly, by adjusting T_M in (4), it is possible to show that the solution proposed in this paper is also robust with respect to packet drop-outs. The same kind of approach appears to be suitable for other problems of control over communication channel with finite data rate, delays and packet drop-out.

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